Nonlinear Inverse Problem for the Estimation of Time-and-Space-Dependent Heat-Transfer Coefficients

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The aim of this paper is to describe a method for the direct estimation of the time-and-space-dependent heat-transfer coefficients from transient temperature data measured at appropriate points inside a heat-conducting solid. This inverse estimation problem is called herein the inverse heat-transfer coefficient problem. An application considered in the present is the quenching of a solid in a liquid. The solution method used here is an extension of the sequential temperature future-information method introduced by Beck for solving the inverse heat-conduction problem. The finite-difference method, based on the control-volume approach, was used for the discretization of the direct heat-conduction problem. Numerical results show that the proposed method is accurate and capable of estimating sudden and large changes in time-and-space-dependent heat-transfer coefficient functions.

Nomenclature

Bi	= Biot number
E	= depth of thermocouples, m
$F(r,\theta)$	= initial temperature distribution, °C
$h(t,\theta)$	= heat-transfer coefficient, W/m²°C
k	= thermal conductivity, W/m°C
L	= number of thermocouples
$\tilde{L}$	= characteristic length, m
M	= number of time steps
P	= number of parameters
r	= number of future time steps
r,θ,ϕ	= spherical coordinate system
R_{in}	= inner radius of sphere, m
R_{out}	= outer radius of sphere, m
t	= time coordinate for inverse problem, s
$\tilde{t}$	= total time, s
$T(r,\theta,t)$	= temperature, °C
$T_\infty(t)$	= fluid temperature, °C
$\mathbf{T}$	= calculated temperature vector, °C
$\mathbf{Y}$	= measurement vector, °C
$\mathbf{Z}$	= sensitivity matrix, °C² m²/W
$\boldsymbol{\beta}$	= parameters vector, W/m² °C
$\varphi_v(\theta)$	= basis functions, Eq. (12)
σ	= standard deviation, °C

I. Introduction

IN convective heat-transfer studies of advanced technological applications with sudden and large changes in convective heat-transfer boundary conditions, values of the transient heat-transfer coefficient $h(t)$ are of primary importance. Typical examples where transient heat-transfer coefficient data are needed include thermal analysis of rocket nozzles and gas-turbine blades, safety analysis of nuclear reactor elements for the loss-of-coolant accident, thermal protection of the Space Shuttle, analysis of quenching processes, and development of new materials and composites for high-temperatures applications. To obtain the required data, one has to measure transient temperatures in an instrumented prototype or scaled model. For bodies with high surface temperatures and subjected to an abrupt change in the convective boundary conditions, it is very difficult to measure directly the surface heat fluxes or the surface temperatures that are required for the estimation of $h(t)$.

In such circumstances, the $h(t)$ can be conveniently estimated from transient temperature measurements at appropriate points inside a heat-conducting solid. This estimation technique depends on solving an inverse problem, the inverse heat-transfer coefficient problem (IHTCP). One-dimensional versions of the IHTCP have been treated previously by Muzzy et al.,[1] Berkovich et al.,[2] and Mehta.[3]

There are two general approaches to the solution of the IHTCP. The first is the "quotient" approach in which the solution of IHTCP is reduced to the solution of the inverse heat conduction problem (IHCP). The surface heat flux and the surface temperature are to be individually estimated, and then the heat-transfer coefficient is determined by the quotient $q/\Delta T$ (Newton's law of cooling). The second approach, called herein the "direct" approach, is to estimate the heat-transfer coefficient directly without the estimation of the intermediate values of the heat flux and the surface temperature. The relative merits of the quotient and direct approaches are discussed in Ref. 14.

The aim of the present paper is to develop a method for solving the two-dimensional IHTCP directly for the time-and-space-dependent heat-transfer coefficient $h = h(t,\theta)$. An application considered here is the quenching of a solid in a liquid, which is frequently necessary in heat treatment. The application is directly applicable to sphere immersed in a liquid and to many other geometries. After the initial transient, steady-state $h(\theta)$ values are obtained. This paper presents the first treatment of the two-dimensional nonlinear IHTCP.

The two well-known methods for stabilizing the solution of the IHTCP are based on the sequential future-information method of Beck[4–7] and the regularization method of Tikhonov.[8,9] Bass et al.,[10] used Beck's method for estimating the heat flux in two-dimensional cylindrical geometry. Irving and Westwater[11] used a variant of Beck's method for obtaining boiling curves. The solution method used here is an extension of the sequential future-information method. This method has been selected because it is more efficient than the "whole domain" regularization method and can be implemented on minicomputers.[9]

This paper is divided into four major sections. The mathematical modeling of the unsteady two-dimensional IHTCP is

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given in Sec. II. In Sec. III the sequential future-information (SFI) method for solving the two-dimensional nonlinear IHTCP is developed. Numerical results of a systematic investigation of the method are given in Sec. IV. The conclusions are given in Sec. V.

II. Statement and Formulation of the Inverse Problem

The physical geometry of a hollow sphere having temperature-independent thermal properties is considered. The sphere is heated and is suddenly dropped in a cold fluid contained in a large vessel. The resultant transient temperature field in the sphere is then axisymmetric with respect to the ϕ direction (refer to the standard spherical coordinate system) and is a function of r and θ only. Because of the symmetry of the temperature field, it suffices to consider the half-sphere shown in Fig. 1 in the formulation of the IHTCP. The inner surface of the sphere is thermally insulated.

The IHTCP for the quenching problem is to find the "best" estimate of the unknown convective heat-transfer coefficient function $h(t,\theta)$ from transient temperature measurements recorded at several points inside the hollow sphere at the same, or different, depths and different θ locations.

The temperature distribution inside the solid body, assuming that the material is homogeneous and isotropic, is described by the heat-conduction equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial T(r,\theta,t)}{\partial r}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T(r,\theta,t)}{\partial\theta}\right]=\frac{1}{\alpha}\frac{\partial T(r,\theta,t)}{\partial t} \tag{1}$$

$$R_{\text{in}} < r < R_{\text{out}}, \qquad 0 < \theta < \pi, \qquad 0 < t \le \tilde{t}$$

subjected to the boundary conditions

$$\frac{\partial T(r,0,t)}{\partial\theta}=\frac{\partial T(r,\pi,t)}{\partial\theta}=0, \qquad R_{\text{in}} \le r \le R_{\text{out}} \tag{2}$$

$$\frac{\partial T(R_{\text{in}},\theta,t)}{\partial r}=0, \qquad 0 \le \theta \le \pi \tag{3}$$

$$-k\frac{\partial T(R_{\text{out}},\theta,t)}{\partial r}=h(t,\theta)\,[T(R_{\text{out}},\theta,t)-T_\infty(t)], \quad 0 \le \theta \le \pi \tag{4}$$

and the initial condition

$$T(r,\theta,0)=F(r,\theta) \tag{5}$$

where α is the thermal diffusivity. The transient heat-transfer coefficient $h(\theta,t)$ in Eq. (4) is the unknown function to be estimated.

The transient temperature histories

$$T(r^*,\theta_\mu,t)=T^*(\theta_\mu,t), \qquad \mu=1,2,\ldots,L, \qquad 0 \le t \le \tilde{t} \tag{6}$$

are known at several interior points at the same radius and different θ locations with coordinates (r^*,θ_μ); $R_{\text{in}} \le r^* < R_{\text{out}}$, $\theta_\mu = (\mu - 1)\,\Delta\theta_L$; $\Delta\theta_L = \pi/L - 1$, $\mu = 1,2,\ldots,L$.

The discrete form of the transient temperature history, Eq. (6), is defined here. The time interval $0 \le t \le \tilde{t}$ is divided into equal subintervals each of length $\Delta t = \tilde{t}/M$ with the discrete time coordinate $t_m = m\Delta t$, $m = 0,1,\ldots,M$. The discrete temperature measurements over time and space are denoted Y and correspond to the continuous function $T^*(\theta_\mu,t)$.

The measurement vector is replaced by

$$Y^t=[Y_{1,0},Y_{2,0},\ldots,Y_{L,0},\ldots,Y_{1,M},Y_{2,M},\ldots,Y_{L,M}] \tag{7}$$

The first and second indices in the subscripts denote space and time, respectively. There are L thermocouples located at the same radius r^* value and different angles θ_μ, $\mu = 1,2,\ldots,L$. In each experiment, there are $M+1$ measurements at L different temperature sensors.

The approximating model for the heat-transfer coefficient $h(t,\theta)$ over the region $(0 \le \theta \le \pi,\ 0 \le t \le \tilde{t})$ involves the following parameters (see Sec. III):

$$\boldsymbol{\beta}^t=[\boldsymbol{\beta}_1^t,\boldsymbol{\beta}_2^t,\ldots,\boldsymbol{\beta}_m^t,\ldots,\boldsymbol{\beta}_M^t] \tag{8a}$$

$$\boldsymbol{\beta}_m^t=[\beta_{1,m},\beta_{2,m},\ldots,\beta_{P,m}] \tag{8b}$$

where P is the number of parameters describing the space variation of the heat-transfer coefficient at each time index m, $m = 1,2,\ldots,M$. The relationship between number of parameters P and number of measurements L is $P \le L$. There must be at least as many sensor locations as there are parameters. The IHTCP is to estimate the heat-transfer coefficient parameters given in Eq. (8), subject to the conditions of Eqs. (1–6).

III. Sequential Future-Information Method

This section presents the SFI method for solving the nonlinear two-dimensional IHTCP. The IHTCP given by Eqs. (1–6) is an ill-posed problem and cannot be solved effectively without using some information concerning the physics of the problem and the structure of the desired solution. The transient temperature response inside the solid body is lagged relative to the surface heat source. This suggests that the estimate of the boundary condition at time t_m from interior temperature data must depend not only on the data up to t_m but also on those for several future time steps.[9] In experiments, measured temperatures are contaminated with noise, which should be treated in the formulation of the solution method in order to reduce the "severe" effects of random errors on the estimated values. The use of temperature future-information and least squares is suitable for the solution of this problem.[5,6]

It is assumed that the vectors $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_{m-1}$ have been estimated one by one, and the task is now to estimate $\boldsymbol{\beta}_m = [\beta_{1,m},\beta_{2,m},\ldots,\beta_{P,m}]'$. In the SFI method, it is necessary to

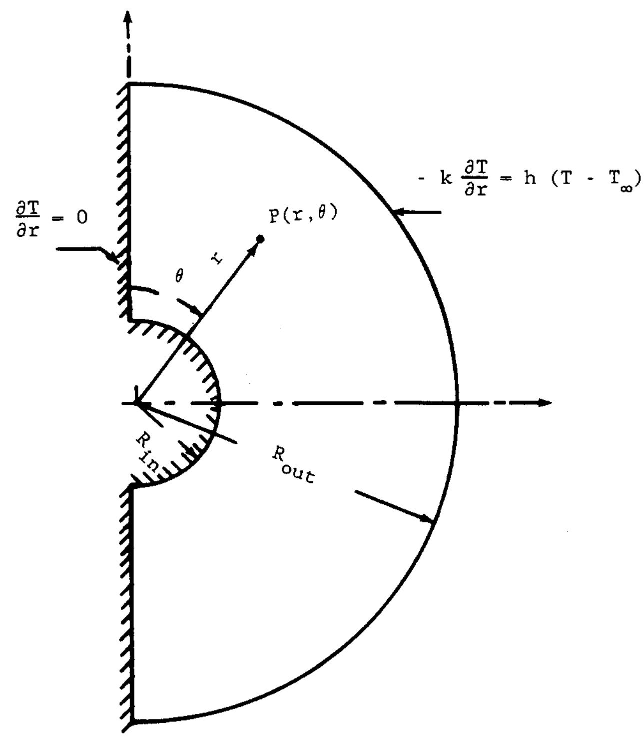

Fig. 1 Solution region for the two-dimensional inverse heat-transfer coefficient problem.

introduce approximations regarding the time and space variation of the unknown function $h(t,\theta)$ in the region $(0 \le \theta \le \pi,\ t_{m-1} \le t \le t_{m+r-1})$, where r is the number of future time steps used at time index m. Consider first the problem of approximating $h(t,\theta)$ over the "analysis" time interval (t_{m-1},t_{m+r-1}). A temporary "assumption" that $h(t,\theta)$ is constant over r future time steps is used:

$$h_{m+1}(\theta) = h_{m+2}(\theta) = \ldots = h_{m+r-1}(\theta) = h_m(\theta) \qquad (9)$$

A simple approach to spatially approximate $h_m(\theta)$ is to divide the interval $0 \le \theta \le \pi$ into $(P-1)$ subintervals or elements of equal length by the points

$$\theta_v = (v-1)\,\Delta\theta_P, \qquad \Delta\theta_P = \frac{\pi}{P-1}, \qquad (v = 1,2,\ldots,P)$$

The $h_m(\theta)$ function is approximated by linear segments between the nodal points θ_v, $v = 1,2,\ldots,P$, as shown in Fig. 2a. In the subinterval $[\theta_v,\theta_{v+1}]$, the appropriate part of the linear approximating function is given by

$$h_m^{(v)}(\theta) = N_v(\theta)\beta_{v,m} + N_{v+1}(\theta)\beta_{v+1,m}, \qquad (\theta_v \le \theta \le \theta_{v+1}) \qquad (10)$$

where

$$N_v(\theta) = \frac{\theta_{v+1} - \theta}{\theta_{v+1} - \theta_v}$$

and

$$N_{v+1}(\theta) = \frac{\theta - \theta_v}{\theta_{v+1} - \theta_v}, \qquad (v = 1,2,\ldots,P-1)$$

The local linear functions $N_v(\theta)$ and $N_{v+1}(\theta)$ are known as shape functions or interpolation functions, and are only defined within an individual element. The values $\beta_{v,m}$ and $\beta_{v+1,m}$ are the unknown nodal parameters.

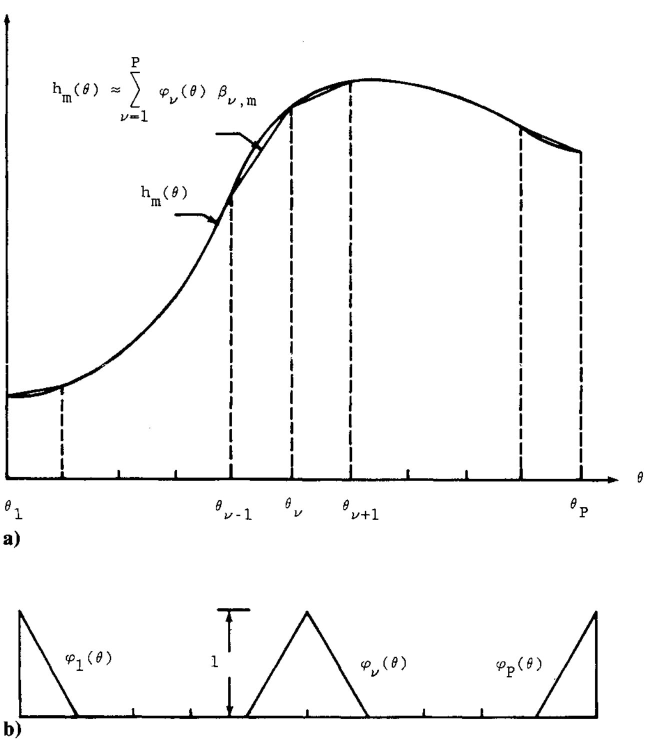

Fig. 2 a) Approximation of $h_m(\theta)$ by linear segments, and b) linear basis functions.

Hence, the global piecewise linear approximating function over the spatial interval $0 \le \theta \le \pi$ may be expressed analytically by a weighted sum as

$$h_m(\theta;\boldsymbol{\beta}_m) \simeq \sum_{v=1}^{P} \varphi_v(\theta)\beta_{v,m} \qquad (11)$$

where the coefficients $\beta_{v,m}$'s, $v = 1,2,\ldots,P$, are the unknown parameters at time t_m; and the $\varphi_v(\theta)$ are a set of linear-basis functions

$$\varphi_v(\theta) = \begin{cases} 1 - \dfrac{|\theta - \theta_v|}{\Delta\theta_P} & |\theta - \theta_v| \le \Delta\theta_P \\ 0 & \text{elsewhere} \end{cases} \qquad (12)$$

The linear-basis functions $\varphi_v(\theta)'s$, which are usually called "tent" or "pyramid" functions, are said to have local support. For example, the function $\varphi_v(\theta)$ is identically zero except for the range $\theta_{v-1} \le \theta \le \theta_{v+1}$ (see Fig. 2b). The local shape functions $N_v(\theta)$ and $N_{v+1}(\theta)$ can be identified as restrictions of the basis functions $\varphi_v(\theta)'s$ to a local element containing nodes v and $v+1$. Cubic splines and other functions could be used instead of linear segments to interpolate the h function at the outer surface. However, higher-order interpolating functions can increase the programming effort in the inverse code.

The SFI method for solving the IHTCP is formulated as the problem of estimating, sequentially over time, the optimal values (in the least-squares sense) of the set of parameters $\beta_{v,m}$, defined in Eq. (8), which minimizes the residuals in the least-squares objective function.

$$S_m^r(\boldsymbol{\beta}_m) = \sum_{\mu=1}^{L}\sum_{i=1}^{r}[Y_{\mu,m+i-1} - T_{\mu,m+i-1}(\boldsymbol{\beta}_m)]^2 \qquad (13)$$

where $Y_{\mu,m+i-1}$ is a temperature measurement, and $T_{\mu,m+i-1}$ is the calculated temperature. The temperature T is a function of the parameter vector $\boldsymbol{\beta}_m$ and is the solution to the associated direct-heat conduction problem obtained by the finite-difference or finite-element methods.

The algebraic expression in Eq. (13) can be conveniently written in matrix notation as

$$S_m^r(\boldsymbol{\beta}_m) = [\boldsymbol{Y} - \boldsymbol{T}(\boldsymbol{\beta}_m)]^t\,[\boldsymbol{Y} - \boldsymbol{T}(\boldsymbol{\beta}_m)] \qquad (14)$$

where $\boldsymbol{Y}$ is a $rL \times 1$ vector defined by

$$\boldsymbol{Y}^t = \{\boldsymbol{Y}_m^t\ \boldsymbol{Y}_{m+1}^t \ldots\ \boldsymbol{Y}_{m+i-1}^t \ldots\ \boldsymbol{Y}_{m+r-1}^t\} \qquad (15a)$$

$$\boldsymbol{Y}_{m+i-1}^t = \{Y_{1,m+i-1}, Y_{2,m+i-1},\ldots,Y_{L,m+i-1}\} \qquad (15b)$$

The vector $\boldsymbol{T}$ is defined similarly.

Taking the gradient of $S_m^r(\boldsymbol{\beta}_m)$ with respect to the vector $\boldsymbol{\beta}_m$, and setting the gradient to zero at the estimated value $\hat{\boldsymbol{\beta}}_m$ of $\boldsymbol{\beta}_m$, one finds that

$$\boldsymbol{Z}^t(\hat{\boldsymbol{\beta}}_m)[\boldsymbol{Y} - \boldsymbol{T}(\hat{\boldsymbol{\beta}}_m)] = \boldsymbol{0} \qquad (16a)$$

where the abbreviation

$$\boldsymbol{Z}^t(\hat{\boldsymbol{\beta}}_m) = \nabla_{\boldsymbol{\beta}_m}\boldsymbol{T}^t(\hat{\boldsymbol{\beta}}_m) \qquad (16b)$$

has been introduced. The system of equations in Eq. (16) are the normal equations for estimating the unknown parameters $\beta_{v,m}, v = 1,2,\ldots,P$, at time index m. The above system is nonlinear because the elements of the sensitivity matrix $\boldsymbol{Z}(\hat{\boldsymbol{\beta}}_m)$ are function of the unknown parameter vector $\hat{\boldsymbol{\beta}}_m$.

One method for solving the normal equations given by Eq. (16) is the Gauss-Newton linearization procedure.[12,13] This method is derived by 1) replacing $\boldsymbol{T}(\hat{\boldsymbol{\beta}}_m)$ in Eq. (16) by its approximate first-order Taylor's expansion about a suitable

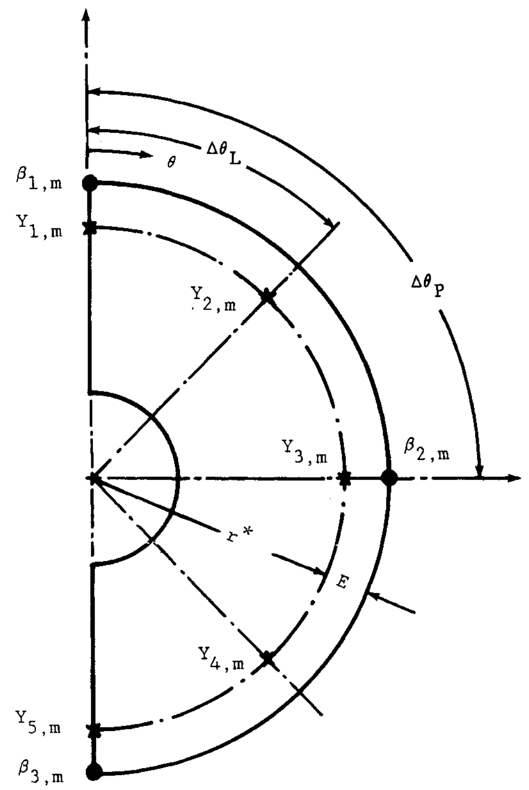

Fig. 3 Layout of the half-sphere showing locations of the thermocouples (*) and the parameters $\beta_{v,m}$ (·).

estimate $\hat{\beta}_m^0$ of the unknown $\hat{\beta}_m$, and 2) approximating $\mathbf{Z}(\hat{\beta}_m)$ in Eq. (16) by $\mathbf{Z}(\hat{\beta}_m) \simeq \mathbf{Z}(\hat{\beta}_m^0)$. If $\hat{\beta}_m^0$ is an estimate of $\hat{\beta}_m$ and $\mathbf{T}(\hat{\boldsymbol{\beta}}_m)$ has bounded partial derivatives about $\hat{\boldsymbol{\beta}}_m^0$, the approximating first-order Taylor's series expansion of $\mathbf{T}(\hat{\boldsymbol{\beta}}_m)$ is given by

$$\mathbf{T}(\boldsymbol{\beta}_m) \simeq \mathbf{T}(\hat{\boldsymbol{\beta}}_m^0) + \mathbf{Z}(\hat{\boldsymbol{\beta}}_m^0)(\hat{\boldsymbol{\beta}}_m - \hat{\boldsymbol{\beta}}_m^0) \tag{17}$$

Using the above approximations in Eq. (16) leads to

$$\hat{\beta}_m = \hat{\beta}_m^0 + [\mathbf{Z}^t(\hat{\beta}_m^0)\mathbf{Z}(\hat{\beta}_m^0)]^{-1}\{\mathbf{Z}^t(\hat{\beta}_m^0)[Y - \mathbf{T}(\hat{\beta}_m^0)]\} \tag{18a}$$

Equation (18a) can be written in a more general iterative form as

$$\hat{\boldsymbol{\beta}}_m^{n+1} = \hat{\boldsymbol{\beta}}_m^n + [\mathbf{Z}^t(\hat{\boldsymbol{\beta}}_m^n)\mathbf{Z}(\hat{\boldsymbol{\beta}}_m^n)]^{-1}\{\mathbf{Z}^t(\hat{\boldsymbol{\beta}}_m^n)[Y - \mathbf{T}(\hat{\boldsymbol{\beta}}_m^n)]\} \tag{18b}$$

which are the linearized equations for the SFI method with n being the iteration index.

The matrix $\mathbf{Z}$ in Eqs.(16–18) is an $rL \times P$ sensitivity matrix (Jacobian matrix) and is given by

$$\mathbf{Z}^{[rL \times P]} = [\mathbf{Z}_m \quad \mathbf{Z}_{m+1} \quad .. \quad \mathbf{Z}_{m+i-1} \quad .. \quad \mathbf{Z}_{m+r-1}]^t \tag{19}$$

where

$$\mathbf{Z}_{m+i-1}^{[L \times P]} = \begin{bmatrix} \dfrac{\partial T_{1,m+i-1}}{\partial \beta_1} & \cdots & \dfrac{\partial T_{1,m+i-1}}{\partial \beta_p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \dfrac{\partial T_{L,m+i-1}}{\partial \beta_1} & \cdots & \dfrac{\partial T_{L,m+i-1}}{\partial \beta_p} \end{bmatrix} \tag{20}$$

The elements of the $\mathbf{Z}_{m+i-1}^{[L \times P]}$ matrix are called the sensitivity coefficients $Z_{\mu,v}(m+i-1) = \partial T_{\mu,m+i-1}/\partial \beta_v$.

In the preceding formulation of the SFI method, the solution of the IHTCP for the unknown vector $\hat{\boldsymbol{\beta}}_m^{n+1}$ at time index m is reduced to that of treating the following problems for every iteration index n:

1) Solving the direct-heat conduction problem for $T_{\mu,m+i-1}(\hat{\boldsymbol{\beta}}_m^n)$ for r future time steps, $i = 1,2,...,r$.

2) Calculation of the sensitivity coefficients $Z_{\mu,v}(m+i-1)$ at the $\hat{\boldsymbol{\beta}}_m^n$ values for r future time steps, $i = 1,2,...,r$.

3) Solution for the system of algebraic equations, Eq. (18), for the unknown parameter vector $\hat{\boldsymbol{\beta}}_m^{n+1}$.

The numerical solution of the direct-heat conduction problem of Sec. II can be accomplished using the finite-difference or finite-element methods. The two-dimensional finite-difference equations can be efficiently solved using an alternating-direction implicit (ADI) method. For more information, see Osman,[14] Bass et al.,[10] and Irving and Westwater.[11]

Sensitivity Coefficients

There are two basic methods for calculating the required derivatives in Eq. (20): the sensitivity equations method and the finite-difference method.[12] The second method is used in the present study. The sensitivity coefficients are approximated by using the forward-difference formula

$$Z_{\mu,v}(m+i-1) = \frac{\partial T_{\mu,m+i-1}}{\partial \beta_v}$$
$$\simeq \frac{T_{\mu,m+i-1}(\beta_1,...,\beta_v + \delta\beta_v,...,\beta_p) - T_{\mu,m+i-1}(\beta_1,...,\beta_v,...,\beta_p)}{\delta\beta_v} \tag{21}$$

where $\delta\beta_v$ is some relatively small quantity such as

$$10^{-5}|\beta_v| \le |\delta\beta_v| \le 10^{-2}|\beta_v|$$

The numerical calculations of the direct problem and the sensitivity coefficients were performed on a VAX-11/750-VMS in double precision. The accuracy of the estimated values obtained for test cases shows that the forward-difference formula given by Eq. (21) is quite satisfactory.

Solution of SFI Equations

The linear system of algebraic equations given by Eq. (18b) was solved for $\hat{\boldsymbol{\beta}}_m^{n+1}$ using the standard Gauss elimination with partial pivoting method.[18] At time index m, the iteration process starts by selecting some initial value $\hat{\boldsymbol{\beta}}_m^0$ and then calculating the corresponding temperatures $T_{\mu,m+i-1}^0$, and the sensitivity coefficients $Z_{\mu,v}^0(m+i-1)$ for $i = 1,2,...,r$. The algebraic system of equations in Eq. (18) is solved for $\hat{\boldsymbol{\beta}}_m^1$. The new estimate $\hat{\boldsymbol{\beta}}_m^1$ is used to replace $\hat{\boldsymbol{\beta}}_m^0$, and the iteration process is repeated until negligible changes occur in the $\hat{\boldsymbol{\beta}}_m$ values. The estimated vector $\hat{\boldsymbol{\beta}}_m$ is retained only over the time interval $t_{m-1} \le t \le t_m$. The time index m is increased by 1, and the solution process is repeated by marching in time until the last vector $\hat{\boldsymbol{\beta}}_M$ is estimated. For more details regarding the solution of Eq. (18), see Refs. 14 and 12.

The SFI method described above is an efficient procedure for solving the IHTCP. If a nonsequential method (e.g., whole domain regularization) were used, it would be necessary to solve a simultaneous system of nonlinear equations of ($M \times P$)th order for the unknown parameters. The SFI method reduces the computations to that of solving a set of M algebraic system each of Pth order. In other words, the SFI method estimates the unknown parameters, in the time-and-space-dependent heat-transfer coefficient, simultaneously-in-space and sequentially-in-time, which results in substantial reduction in computation time and memory.

IV. Numerical Results and Discussions

This section contains the numerical results of an investigation of the SFI algorithm for some selected IHTCP test cases. The numerical experiments are designed to illustrate the following aspects of the proposed SFI method:

1) Ability of the proposed procedure to estimate sudden

and large changes in time-and-space-dependent convective heat-transfer coefficient functions.

2) Reduced sensitivity to random errors in the input temperature data.

3) Effects of the future-temperature information on stability and accuracy of the solution of the IHTCP.

The numerical experiments are conducted for a copper sphere in a quenching problem. The following data are used in the numerical modeling:

Inner radius $R_{in} = 0.01$ m

Outer radius $R_{out} = 0.03$ m

$$F(r,\theta) = T_0 = 99.0°C$$

$$k = 379.0 \text{ W/m°C}$$

$$\alpha = 11.234\ 10^{-5} \text{ m}^2/\text{s}$$

$$T_\infty(t) = T_\infty = 20.0°C$$

Five thermocouples are located at internal points having the same radius $r^* = 0.024$ m and the angular locations $\theta = 0$, $\pi/4$, $\pi/2$, $3\pi/4$, and π rad. The number of unknown parameters describing the space variation of h_m is 3; at the angular locations $\theta = 0, \pi/2$, and π rad (see Fig. 3).

The two-dimensional transient direct-heat conduction problem in a hollow sphere was solved by using the finite-difference method. The formulation of the finite-difference equations is based on the finite control-volume (FCV) technique.[9,15] Approximate FCV equations are obtained by discretizing the spatial domain ($R_{in} \le r \le R_{out}$, $0 \le \theta \le \pi$) into mesh points (nodes) having discretization coordinates (r_i, θ_j) with $r_i = i\Delta r + R_{in}$, $i = 1,2,\ldots,I$, and $\theta_j = j\Delta\theta_J$, $j = 1,2,\ldots,J$. The Δr and $\Delta\theta_J$ are given by

$$\Delta r = [(R_{out} - R_{in})/I]$$

and

$$\Delta\theta_J = \pi/J$$

The control-volume surfaces are located midway between the grid points. Simple energy balances are performed over the control volumes surrounding each node or grid point.

The FCV equations are solved by the ADI scheme.[16,17] The ADI scheme is an efficient procedure for solving two-dimensional transient heat-conduction problems. The advantage of the ADI method over fully implicit methods is that the resultant systems of algebraic equations have a tridiagonal coefficient matrices that can be solved efficiently by Thomas' algorithm. In the ADI scheme, two systems of difference equations are used in turn over successive time steps each of size $\Delta t/2$. The first system of equations is implicit in the r direction and explicit in the θ direction. The second system of equations is implicit in the θ direction and explicit in the r direction. The details of the formulation of the finite control-volume equations and ADI scheme are available elsewhere.[14]

The test case numerically investigated is one in which the heat-transfer coefficient varies in time in a triangular fashion and linear in space. The values of the heat-transfer coefficients considered cover the range of Biot number Bi from lumped body to boiling heat transfer. The Bi is defined by

$$Bi = h\tilde{L}/k \tag{22a}$$

$$\tilde{L} = \frac{\text{volume of sphere}}{\text{outer surface area}} = \frac{R_{out}^3 - R_{in}^3}{3R_{out}^2} \tag{22b}$$

where $\tilde{L}$ is a characteristic length for the sphere geometry.

Two FORTRAN V computer programs were written to study the IHTCP test cases. Program NUMTWO was written to solve the two-dimensional unsteady direct-heat conduction problem for the quenching of a spherical body with a specified heat-transfer coefficient function $h(t,\theta)$. This program implements the ADI scheme. Program NLINV1 was written to solve the IHTCP. The NLINV1 program utilizes the SFI method described in Sec. III, and uses NUMTWO as a subroutine.

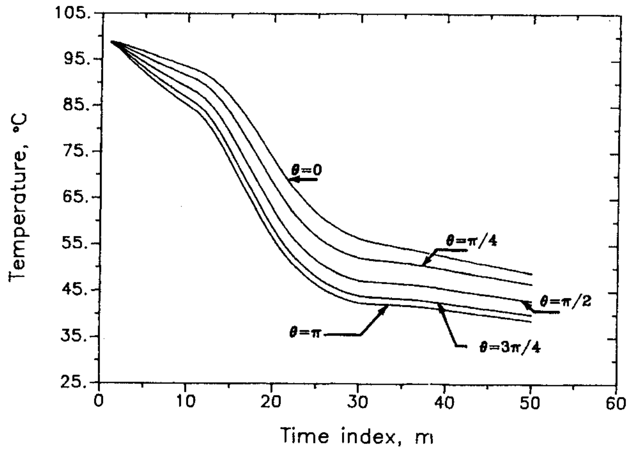

Fig. 4 Simulated temperature histories at thermocouple locations.

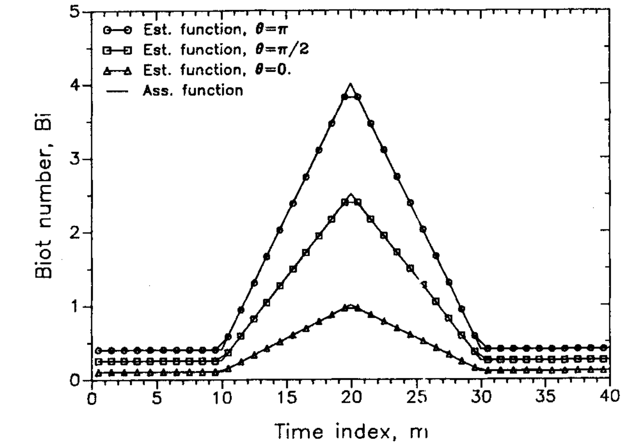

Fig. 5 Estimated $Bi(t,\theta)$ function obtained by using the SFI method with parameter values $\Delta t_E^+ = 0.156$, $\sigma = 0°C$, $r = 1$.

The direct-heat conduction problem with the triangular pulse heat-transfer coefficient was solved numerically to generate temperature test data for the IHTCP method. Five temperature profiles at the thermocouple locations and for the time interval $0 \le \tau \le 2.5$ s were computed numerically using 11 nodes in the r direction, 9 nodes in the θ direction, and $\Delta\tau = 0.05$ s. The five temperature profiles are shown in Fig. 4.

Figure 5 shows both the assumed Bi function and estimated Bi function obtained by using the SFI algorithm for the triangular pulse test case with errorless data ($\sigma = 0$) and for no additional future information (i.e., $r = 1$). The nondimensional time based on the thermocouples depth E, $\Delta t_E^+ = \alpha\Delta t/E^2$ is 0.156. The reconstruction of the Bi function is excellent in this test case since it is nearly indistinguishable from the assumed function.

Figure 6 demonstrates the accuracy and variability of the estimated functions when $\sigma = 0.005°C$ and $r = 1$. Here the three plots a, b, and c depict both the assumed function and the estimated function at the bottom ($\theta = \pi$), equator ($\theta = \pi/2$), and top ($\theta = 0$) of the sphere, respectively. Although the standard deviation σ is very small, the recovery of the assumed function is very poor at all the three locations. In part (a), for Bi at the bottom of the sphere, the oscillations start at the 15th time step. However, the solution is stable in the time

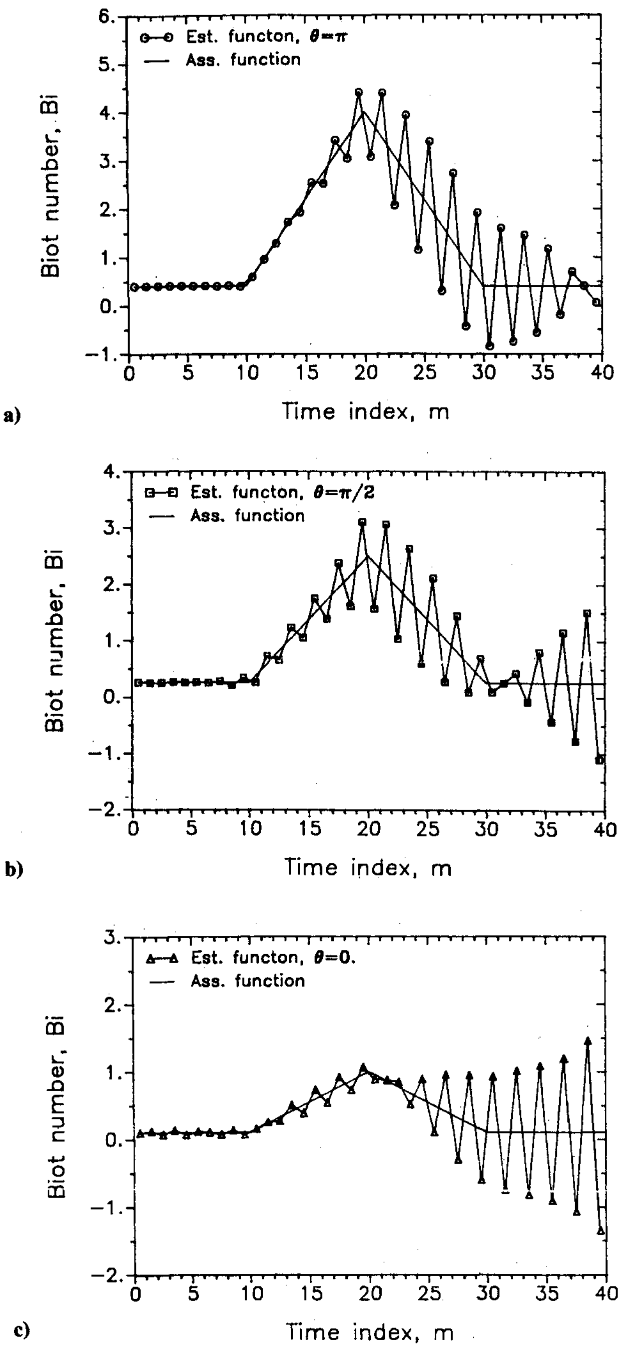

Fig. 6 Estimated $Bi(t,\theta)$ function at: a) $\theta = \pi$, b) $\theta = \pi/2$, and c) $\theta = 0$; $\Delta t_E^+ = 0.156$, $\sigma = 0.005°C$, $r = 1$.

range considered. In part (b) the oscillations start as early as the 10th time step, and after the 34th time step the solution becomes unstable. In part (c) the oscillations are even worse, and the instability starts at about the 25th time step. The conclusion is that for no additional future temperatures, the estimation of very small Bi values may cause some difficulties especially with small dimensionless time step and "large" errors in the data.

Figure 7 illustrates the effects of the temperature future-information on the accuracy and stability of the solution of the IHTCP. Four future time steps ($r = 4$) are used in this case with $\sigma = 0.25°C$ (i.e., errors are 50 times, on the average, larger than those for Fig. 6). This value of σ simulates the accuracy of actual data acquisition systems. In Fig. 7 there are no oscillations or instability observed, and the agreement between the assumed function and the estimated one is very

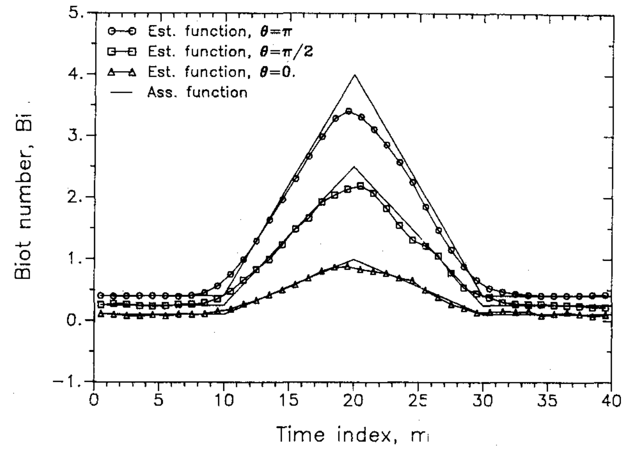

Fig. 7 Estimated $Bi(t,\theta)$ function with $\Delta t_E^+ = 0.156$, $\sigma = 0.25°C$, $r = 4$.

good. The good results cannot be attributed to the use of triangular-in-time test case because the time basis functions in the algorithm are not linear in time; it is linear in space only. The use of future time steps ($r = 4$) greatly reduces the sensitivity to errors. The estimated results are very accurate except in the neighborhood of the triangle corners where the effects of some smoothing are visible over two time steps (0.1 s). For the triangular-in-time heat transfer coefficient test case and with a standard deviation of $\sigma = 0.25°C$, a value of $r = 4$ was found to be "optimal". Values of $r = 2$ or 3 produced some oscillations, while values of $r > 4$ produced excess smoothing in the neighborhood of the triangle corners. The SFI method gives a biased estimate of the heat transfer function while very substantially reducing the sensitivity to errors.

V. Conclusions

A sequential inverse method is described for the estimation of the time-and-space-dependent heat-transfer coefficients when the transient temperature histories at appropriate interior points inside the heat-conducting solid are available. The numerical results of the IHTCP test cases show that the SFI method is accurate and able to handle abrupt and steep large changes in the heat-transfer coefficients in two-dimensional problems.

The SFI estimation procedure described here is quite general and can be extended easily to cylindrical and rectangular coordinate systems, and three-dimensional problems. It is recommended for the experimental estimation of the transient heat-transfer coefficients in applications such as quenching processes and short-duration wind-tunnel experiments.

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